

Orbital Functional Geometry

Orbits, Spectra, and the Geometry of Analytic Functions

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Abstract

We introduce the *Orbital Space* $\mathcal{O}$, a new mathematical object constructed from the equation $e^{P(x)(az+b)} = f(x)$, where P is a polynomial, f an analytic function, and $(a, b) \in \mathbb{R}^* \times \mathbb{R}$. A point of $\mathcal{O}$ is a closed orbit — the trajectory of the solution z as x makes one full turn around a root of P in $\mathbb{C}$.

We develop the complete theory of $\mathcal{O}$: metric structure (Hausdorff distance on admissible orbits), foliation, symmetries, and a classification into 2^{n+1} classes for P of degree n . We establish an encoding theorem showing that the zero-pole structure of f is fully readable from the geometry of orbits, and introduce orbital spectroscopy — a method to count and locate zeros of analytic functions without solving equations.

A differential and integral calculus is developed in three directions (a, b, x_0) with qualitatively distinct natures: linear, logarithmic, and double-logarithmic. The orbital Laplacian $\Delta_{\mathcal{O}}$ is defined; its continuous spectrum is $\lambda = 2/a^2$ with eigenfunctions $f = e^{\alpha x + \beta}$. The intrinsic geometry of $\mathcal{O}$ is shown to be flat, with all curvature extrinsic and induced by f .

Several directions remain open: discrete spectrum, full consequences of the Hausdorff metric, and classification of orbital flows.

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Introduction

Classical analysis studies functions by their values, derivatives, and zeros. The ambient space is always fixed — $\mathbb{R}$, $\mathbb{C}$, or some function space. The function moves through the space; the space itself does not depend on the function.

This paper takes a different starting point. Given a polynomial P , an analytic function f , and parameters (a, b) , we ask: what space does the equation

$$e^{P(x)(az+b)} = f(x)$$

generate? Here $az + b$ is not time. It is the line of an independent function — a structural replacement for the free variable in the exponent.

The solution

$$z = \frac{1}{a} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

has a remarkable property: when x makes a full turn around a root x_0 of P in $\mathbb{C}$, the trajectory of z closes into an orbit. This closed orbit is a *point* of the space $\mathcal{O}$.

What emerges is not a transformation of f , nor a representation. It is a space *generated by* f — where the analytic structure of f (its zeros, poles, logarithmic derivative) becomes geometric structure (curvature, spectral jumps, orbital deformation).

The theory developed here is complete in its foundations and open in several directions. It grew, unexpectedly, from an attempt to study the Riemann zeta function. That attempt produced a different object entirely.

1 The Fundamental Object

Definition 1 (Orbital equation). *Let $P(x)$ be a polynomial and $f(x)$ a function with $\text{roots}(P) \cap \text{roots}(f) = \emptyset$. Let $g(z) = az + b$ be the line of an independent function, $a \in \mathbb{R}^*$, $b \in \mathbb{R}$.*

The orbital equation is:

$$e^{P(x) \cdot (az+b)} = f(x)$$

Here $az + b$ is not time. It is the variation driven by the line of an independent function.

Remark 1 (Natural extension to analytic P). *The definition uses P polynomial for concreteness — roots are isolated, their structure is explicit, and the classification into 2^{n+1} classes is combinatorially clear.*

However, the theory does not depend on P being polynomial. Any analytic function P with isolated roots works: $P = \sin x$ gives a periodic foliation of period π with infinitely many classes; $P = \ln x$ gives $z = \log_x f(x)$, a dynamic exponent.

The polynomial case is the base framework. Analytic P is the natural extension — the structure of $\mathcal{O}$ transfers without modification.

Lemma 1 (Mixed solutions). *Taking logarithms:*

$$z = \frac{1}{a} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

- $f(x) > 0$: z is real
- $f(x) < 0$: z is complex, with imaginary part $\frac{\pi}{aP(x)}$

Real and complex solutions coexist in the same space.

2 Structure of the Space

A *point* of $\mathcal{O}$ is an orbit — the closed trajectory of z when x makes one full turn around a root x_0 of P in $\mathbb{C}$:

$$x(\theta) = x_0 + r \cdot e^{i\theta}, \quad \theta \in [0, 2\pi]$$

for fixed (a, b) and radius $r > 0$.

A point is identified by the triple (a, b, x_0) : a controls scale and direction, b controls the position of centres, x_0 is the root of P around which the orbit closes.

The *metric* between two orbits C_1 and C_2 is the Hausdorff distance:

$$d_H(C_1, C_2) = \max \left(\sup_{z \in C_1} \inf_{w \in C_2} |z - w|, \sup_{w \in C_2} \inf_{z \in C_1} |z - w| \right)$$

The earlier pointwise metric $|\ln f - \ln g|$ is a local approximation, not the metric of $\mathcal{O}$.

The *Euler constraint* $e^{2\pi i} = 1$ forces the rotation to close. Fixed points are the roots of $P(x)$.

3 Example: $P(x) = x$, $f(x) = x$

$$z = \frac{1}{a} \left(\frac{\ln x}{x} - b \right)$$

The function $\frac{\ln x}{x}$ has a unique maximum at $x = e$. The constant e emerges as a bifurcation: two real centres if $b < \frac{1}{e}$, one if $b = \frac{1}{e}$, none if $b > \frac{1}{e}$.

When $x < 0$: z acquires imaginary part $\frac{i\pi}{ax}$. The constant π appears naturally on the complex side.

Neither e nor π is imposed. Both emerge.

4 Topology for Degree n

Let $P(x)$ have n real roots $x_1 < x_2 < \dots < x_n$. These create $n + 1$ zones in $\mathbb{R}$.

Lemma 2 (Alternating sign). *The sign of z alternates between consecutive zones. If zone k has $z > 0$, then zone $k + 1$ has $z < 0$. This is forced by $P(x)$ changing sign at each root.*

The topology of $\mathcal{O}$ depends on two polynomials simultaneously:

- Roots of $P(x)$ create the zones.
- Roots of $f(x)$ decide which zones are real or complex.

This is a two-level topology: the first level is structural (given by P), the second is functional (given by f).

Example $n = 3$: $P(x) = x^3 - x$, roots at $-1, 0, 1$. With $f(x) = x^2 + 1 > 0$ always, all zones are real and z alternates sign: negative, positive, negative, positive across the four zones.

5 Extension to $x \in \mathbb{C}$

When $x = u + iv \in \mathbb{C}$, the logarithm becomes:

$$\ln f(x) = \ln |f(x)| + i \arg(f(x))$$

The solution z is always complex. The real/complex distinction of the real case is replaced by the *branch* of the logarithm.

The roots of $P(x)$ in $\mathbb{C}$ are isolated points in the plane — no longer boundaries cutting a line, but centres of rotation in a plane.

Lemma 3 (Automatic Euler constraint). *When x winds around a root of P in $\mathbb{C}$, the logarithm gains $2\pi i$:*

$$\ln f(x) \rightarrow \ln f(x) + 2\pi i$$

The Euler constraint $e^{2\pi i} = 1$ is no longer imposed. It emerges automatically from the geometry of $\mathbb{C}$.

This is a key difference between the real and complex cases: in $\mathbb{R}$, the rotation constraint is forced. In $\mathbb{C}$, it is natural.

6 Branch Structure in $\mathbb{C}$

The complex logarithm is multi-valued:

$$\ln f(x) = \ln |f(x)| + i(\arg(f(x)) + 2\pi k), \quad k \in \mathbb{Z}$$

Each k gives a distinct branch — a distinct orbit in $\mathcal{O}$:

$$z_k = \frac{1}{a} \left(\frac{\ln |f(x)| + i(\arg(f(x)) + 2\pi k)}{P(x)} - b \right)$$

Lemma 4 (Distance between branches). *The distance between consecutive branches is:*

$$d(z_k, z_{k+1}) = \frac{2\pi}{|a \cdot P(x)|}$$

Near a root of P : branches diverge to infinity. Far from roots: branches converge toward zero. The roots of P organise the hierarchy of branches.

Lemma 5 (Euler constraint as branch metric). *The unit of separation between branches is 2π . This is the same 2π as in $e^{2\pi i} = 1$.*

The Euler constraint and the branch metric are the same object:

- *Euler: a full rotation returns to 1*
- *Branches: two consecutive orbits are separated by 2π*

The rotation constraint is not external. It is the metric of the branches itself. The space closes on itself naturally.

7 Foliation of $\mathcal{O}$

Lemma 6 (Branches never cross). *Two distinct branches z_k and z_j with $k \neq j$ satisfy $z_k \neq z_j$ everywhere. The orbits of $\mathcal{O}$ are disjoint.*

Proof. $z_k = z_j$ requires $\frac{2\pi k}{P(x)} = \frac{2\pi j}{P(x)}$, which forces $k = j$. □

Lemma 7 (Non-uniform foliation). *$\mathcal{O}$ is foliated: it consists of infinitely many disjoint sheets $k \in \mathbb{Z}$, each a copy of the orbital space for a fixed branch.*

The distance between sheets k and $k+1$:

$$d(z_k, z_{k+1}) = \frac{2\pi}{|a \cdot P(x)|}$$

is not uniform. The sheets are not parallel — they open at the roots of P (distance $\rightarrow \infty$) and close far from roots (distance $\rightarrow 0$).

The foliation is controlled entirely by $P(x)$.

The structure was not constructed. It emerged from the single condition that branches do not cross.

8 Reflection Symmetry and the Role of a

Lemma 8 (Reflection symmetry). *Under $a \rightarrow -a$, the orbits satisfy $z_k^{(-a)} = -z_k^{(+a)}$. The distances between sheets are unchanged — $|a|$ removes the sign. But the order of sheets is reversed: the foliation is reflected as in a mirror.*

The map $a \leftrightarrow -a$ is a fundamental symmetry of $\mathcal{O}$: it reverses orientation without changing structure.

Lemma 9 (Why $a \neq 0$). *At $a = 0$, all sheets diverge simultaneously. The foliation collapses. The space ceases to exist.*

The condition $a \in \mathbb{R}^$ is not arbitrary — it is the condition for $\mathcal{O}$ to exist. The two orientations $a > 0$ and $a < 0$ are two mirror images of the same space, separated by the void $a = 0$.*

9 Symmetry Group of $\mathcal{O}$

Lemma 10 (Translation symmetry). *Under $b \rightarrow -b$:*

$$z_k^{(-b)} = z_k^{(+b)} + \frac{2b}{a}$$

All sheets translate together by $\frac{2b}{a}$. Distances between sheets are unchanged. The foliation is preserved — only the position changes.

Lemma 11 (Two fundamental symmetries). *$\mathcal{O}$ admits two fundamental symmetries:*

- $a \leftrightarrow -a$ — reflection: reverses orientation
- $b \leftrightarrow -b$ — translation: shifts position

Together they generate the group of isometries of $\mathbb{R}$ — the infinite dihedral group D_∞ .

These symmetries were not constructed. They emerged from the parameters (a, b) .

10 Rotation in x as Translation in $\mathcal{O}$

Lemma 12 (Rotation becomes translation). *When x rotates by angle θ around a root x_0 of P :*

$$x \rightarrow x_0 + (x - x_0)e^{i\theta}$$

the orbit transforms as:

$$z_k \rightarrow z_k + \frac{i\theta}{aP(x)}$$

A rotation in x is a translation between sheets in $\mathcal{O}$.

Lemma 13 (Quantisation of sheets). *For a full rotation $\theta = 2\pi$:*

$$z_k \rightarrow z_k + \frac{2\pi i}{aP(x)} = z_{k+1}$$

A complete rotation in x moves exactly one sheet forward in $\mathcal{O}$. k rotations move k sheets. The index $k \in \mathbb{Z}$ counts rotations.

The sheets are indexed by $\mathbb{Z}$ because rotations are quantised: only full turns return to the same point. The quantisation of sheets comes from the quantisation of rotations.

11 Fundamental Invariant and Classification

Lemma 14 (Fundamental invariant). *The distance between consecutive sheets:*

$$d(z_k, z_{k+1}) = \frac{2\pi}{|a \cdot P(x)|}$$

is invariant under all three symmetries of $\mathcal{O}$:

- $a \leftrightarrow -a$: depends on $|a|$, not a
- $b \leftrightarrow -b$: independent of b
- Rotation in x : $P(x)$ unchanged around its roots

The inter-sheet distance is the fundamental invariant of $\mathcal{O}$. It is controlled entirely by $P(x)$.

Lemma 15 (Classification by P). *$P(x)$ is the identity of the space. Two orbital spaces built with the same P have the same fundamental geometry, regardless of their parameters (a, b) .*

Two orbital spaces are equivalent if and only if they have the same P up to a constant. $P(x)$ is the fingerprint of $\mathcal{O}$.

12 The Affine Group of $\mathcal{O}$

Lemma 16 (Transition between equivalent spaces). *Two orbital spaces with the same P and parameters (a_1, b_1) , (a_2, b_2) are related by:*

$$z^{(2)} = \alpha \cdot z^{(1)} + \beta$$

where $\alpha = \frac{a_1}{a_2}$ and $\beta = \frac{b_1 a_2 - b_2 a_1}{a_1 a_2}$.

This is an affine transformation — multiplication and translation. The set of all such transformations forms the affine group of $\mathbb{C}$.

Remark 2 (The line is everywhere). *The fundamental object of $\mathcal{O}$ is the line $az + b$. The group of symmetries between equivalent spaces is the affine group — generated by lines.*

The object that founds the space is also its symmetry group. This is a deep coherence of $\mathcal{O}$.

13 Classification by (P, f)

The full classification of $\mathcal{O}$ requires both P and f .

Level 1 — Geometric class: two spaces are in the same geometric class if they have the same P . This fixes the foliation, distances, and symmetries.

Level 2 — Nature pattern: within a geometric class, f determines which zones are real ($f > 0$) or complex ($f < 0$).

Lemma 17 (Counting classes). *For $P(x)$ of degree n with n real roots, there are $n + 1$ zones. Each zone has two possible natures (+ or −). The number of distinct orbital spaces in the same geometric class is:*

$$2^{n+1}$$

Each class is a pattern — a sequence of + and − of length $n + 1$.

Example: $P(x) = x(x - 1)$, three zones $(-\infty, 0)$, $(0, 1)$, $(1, +\infty)$. The 8 patterns:

$(+, +, +)$, $(+, +, -)$, $(+, -, +)$, $(+, -, -)$, $(-, +, +)$, $(-, +, -)$, $(-, -, +)$, $(-, -, -)$

Each pattern is a distinct orbital space with the same geometry but different real/complex structure.

14 Duality in $\mathcal{O}$

Among the 2^{n+1} patterns, two are extremal:

All-positive pattern $(+, +, \dots, +)$: $f > 0$ everywhere. z is always real. Branches $k \neq 0$ exist but are purely imaginary — decoupled from the real structure. The foliation is entirely real. Branches are hidden.

All-negative pattern $(-, -, \dots, -)$: $f < 0$ everywhere. z is always complex with imaginary part $\frac{\pi}{aP(x)}$. All branches are active. The foliation is entirely complex.

Lemma 18 (Fundamental duality). *The two extremal patterns are dual under $f \rightarrow -f$:*

$$all\ real \longleftrightarrow all\ complex$$

$$branches\ hidden \longleftrightarrow branches\ active$$

Changing the sign of f reverses the duality. Mixed patterns lie between the two extremes — some zones real, others complex, some branches active, others hidden.

15 Beyond Polynomials: General P

The construction of $\mathcal{O}$ requires only that P have isolated roots and be analytic. Polynomials satisfy this, but they are not the only choice.

Case $P(x) = \sin(x)$: roots at $x = k\pi$, $k \in \mathbb{Z}$. Infinitely many boundaries, infinitely many zones of length π . The inter-sheet distance:

$$d(z_k, z_{k+1}) = \frac{2\pi}{|a \cdot \sin(x)|}$$

opens and closes with period π — the foliation *breathes*. The constant π appears now as the period of the foliation, not only in complex branches.

The classification explodes: 2^∞ classes, indexed by infinite sequences of $+$ and $-$. Special classes include periodic sequences $(+, -, +, -, \dots)$, constant sequences, and arbitrary ones.

Case $P(x) = \ln x$: one root at $x = 1$. The solution becomes:

$$z = \frac{1}{a} \left(\frac{\ln f(x)}{\ln x} - b \right) = \frac{1}{a} (\log_x f(x) - b)$$

A point in $\mathcal{O}$ is now an *exponent* — the power of x needed to reach $f(x)$.

Example: $f(x) = x^n$ gives $z = n$ everywhere — a constant point in $\mathcal{O}$. The distance between $z = 2$ and $z = 3$:

$$d = |\ln x^2 - \ln x^3| = |\ln x|$$

depends on x . The geometry is *dynamic* — it changes with position. The boundary at $x = 1$ is where the base of the logarithm becomes undefined — a natural, not arbitrary, limit.

16 Generalising $g(z)$: Beyond the Line

The fundamental object $e^{P(x) \cdot g(z)} = f(x)$ was constructed with $g(z) = az + b$ — a line. But g can be any analytic function.

Case $g(z) = z^2$:

$$z = \pm \sqrt{\frac{\ln f(x)}{P(x)}}$$

Two solutions for each x . The space *doubles*. Nature depends on the sign of $\frac{\ln f(x)}{P(x)}$: real pair if positive, imaginary pair if negative, single point $z = 0$ if zero.

Lemma 19 (Multiplicity from degree of g). *For $g(z) = z^m$, the orbital equation has m solutions:*

$$z_j = \left(\frac{\ln f(x)}{P(x)} \right)^{1/m} \cdot e^{2\pi i j / m}, \quad j = 0, 1, \dots, m-1$$

The m solutions form an orbit with m branches, symmetric under the cyclic group $\mathbb{Z}/m\mathbb{Z}$.

The degree of g controls the multiplicity of points in $\mathcal{O}$.

The full classification of $\mathcal{O}$ requires three levels:

- P — geometric class (foliation, distances)
- f — nature pattern (real/complex zones)
- $\deg(g)$ — multiplicity (number of branches per point)

17 Complex P : Directional Foliation

When P has complex coefficients, its roots lie anywhere in $\mathbb{C}$. Each root $x_0 = re^{i\theta}$ carries two pieces of information:

- $|x_0|$ — where the foliation opens
- $\arg(x_0)$ — the direction in which it opens

Example: $P(z) = z^2 + 1$, roots at $\pm i$. The symmetry $z \leftrightarrow \bar{z}$ exchanges the two roots. $\mathcal{O}$ is its own mirror image under conjugation. The distance between the two boundaries $|i - (-i)| = 2$ is real — real geometry emerges from complex boundaries.

Lemma 20 (Directional foliation). *For P of degree n with n complex roots, the foliation has n opening directions — one per root, determined by its argument. The foliation is directional in $\mathbb{C}$.*

For P with real coefficients: complex roots come in conjugate pairs. Directions cancel in pairs. The foliation remains symmetric. Real P is the case where all directions balance.

The full classification now requires:

- P — geometric class
- f — nature pattern
- $\deg(g)$ — multiplicity
- $\arg(\text{roots of } P)$ — opening directions

18 Product of Two Orbital Spaces

Given two orbital spaces with the same $g(z) = az + b$:

$$e^{P_1(x) \cdot (az+b)} = f_1(x), \quad e^{P_2(x) \cdot (az+b)} = f_2(x)$$

their product is defined by multiplying the equations:

$$e^{(P_1(x)+P_2(x)) \cdot (az+b)} = f_1(x) \cdot f_2(x)$$

A new orbital space with $P = P_1 + P_2$ and $f = f_1 \cdot f_2$.

Lemma 21 (Centering property). *For $P_1(x) = x - x_1$ and $P_2(x) = x - x_2$:*

$$P_1 + P_2 = 2x - (x_1 + x_2)$$

with root at $x = \frac{x_1+x_2}{2}$ — the midpoint.

The structural boundary of the product lies exactly between the two original boundaries. The product always centres.

Remark 3 (Exchange of roles). *In the product, the original structural boundaries x_1 and x_2 become roots of $f = f_1 \cdot f_2$ — they become functional boundaries. What was structural becomes functional. A new structural boundary appears at the centre.*

19 Convergence of Orbits

Lemma 22 (Rigidity in b). *Two orbits with parameters (a, b_1, x_0) and (a, b_2, x_0) satisfy:*

$$|z_1(\theta) - z_2(\theta)| = \frac{|b_1 - b_2|}{|a|}$$

The distance is constant along the entire orbit. Two orbits close in b remain parallel — they neither converge nor diverge. $\mathcal{O}$ is isometric in the b direction.

Lemma 23 (Flexibility in a). *Two orbits with parameters (a_1, b, x_0) and (a_2, b, x_0) satisfy:*

$$|z_1(\theta) - z_2(\theta)| = \left| \frac{1}{a_1} - \frac{1}{a_2} \right| \cdot \left| \frac{\ln f(x(\theta))}{P(x(\theta))} - b \right|$$

The distance varies with θ — it depends on the position along the orbit. Two orbits close in a approach and separate as they turn. $\mathcal{O}$ is flexible in the a direction.

$\mathcal{O}$ has two natures of convergence: rigid (isometric) in b , flexible (variable) in a .

20 Convergence in x_0 : Role of the Logarithmic Derivative

Two orbits with nearby roots $x_0^{(1)} = 0$ and $x_0^{(2)} = \epsilon$ (small), same $P_1(x) = x$, $P_2(x) = x - \epsilon$, same (a, b) . For small ϵ :

$$|z_1(\theta) - z_2(\theta)| \approx \frac{\epsilon}{|a| \cdot r} \cdot \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|$$

The distance between two nearby orbits is proportional to the *logarithmic derivative* of f evaluated on the orbit:

$$(\ln f)' = \frac{f'}{f}$$

Lemma 24 (Three natures of convergence). $\mathcal{O}$ has three distinct convergence behaviours:

- In b : **rigid** — constant distance, parallel orbits
- In a : **flexible** — variable distance, orbits breathe
- In x_0 : **functional** — distance controlled by the logarithmic derivative of f

Each parameter governs a different geometry of proximity.

21 Curvature of $\mathcal{O}$: Geometry Charged by f

Integrating the inter-orbit distance over a full turn:

$$\oint |z_1 - z_2| d\theta \approx \frac{\epsilon}{|a| \cdot r} \oint \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta$$

By the argument principle:

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = N - P$$

where N = zeros of f inside the orbit, P = poles.

Lemma 25 (Charged geometry). Zeros of f inside an orbit increase the integrated inter-orbit distance — they repel nearby orbits. Poles of f decrease it — they attract nearby orbits.

$\mathcal{O}$ has a charged geometry: zeros of f are sources of repulsion, poles are sources of attraction. Orbits deform around these charges.

Remark 4 (Roles of P and f). The proximity of orbits in $\mathcal{O}$ depends on both P and f simultaneously — through their logarithmic derivatives $(\ln f)'$ and $(\ln P)'$. The separation global/P versus local/f is too simple:

- P — structure of the space (foliation, boundaries, classification) and part of the proximity geometry
- f — content of the space (zeros, poles, curvature) and the other part of proximity

The geometry of proximity requires both together.

22 Argument Principle in $\mathcal{O}$

Lemma 26 (Orbits encode zeros and poles). When x completes one full turn around x_0 , the variation of z is:

$$\Delta z = z(2\pi) - z(0) = \frac{2\pi i(N - P)}{a \cdot P(x_0)}$$

where N = zeros and P = poles of f inside the orbit.

The displacement of z after one full turn directly encodes the zero-pole balance of f .

Lemma 27 (Reading f from orbits). • $\Delta z = 0$: f has equal zeros and poles inside

• $\Delta z \neq 0$: imbalance between zeros and poles

• $|\Delta z|$ large: many zeros or poles inside

The argument principle of complex analysis becomes a geometric statement in $\mathcal{O}$: the zero-pole structure of f is readable directly from the displacement of orbits.

23 Complete Reading of f from Orbits

Lemma 28 (Velocity of an orbit). The velocity of z along an orbit is:

$$\frac{dz}{d\theta} = \frac{1}{a} \cdot \frac{d}{d\theta} \left(\frac{\ln f(x(\theta))}{P(x(\theta))} \right)$$

When f has no zeros or poles inside: the velocity is regular, no peaks. When f has zeros and poles inside: the velocity has peaks — accelerations near poles, decelerations near zeros.

The regularity of $\frac{dz}{d\theta}$ encodes $N + P$.

Theorem 1 (Complete encoding). Two quantities read from a single orbit determine N and P separately:

$$N - P = \frac{a \cdot P(x_0)}{2\pi i} \cdot \Delta z \quad N + P \text{ from regularity of } \frac{dz}{d\theta}$$

$$N = \frac{(N - P) + (N + P)}{2}, \quad P = \frac{(N + P) - (N - P)}{2}$$

The number of zeros and poles of f inside any orbit is completely determined by the geometry of that orbit.

$\mathcal{O}$ encodes complex analysis geometrically. It is not only a metric space — it is a space where the analytic structure of f is readable from orbits.

24 Localisation of Zeros via Orbits

$\mathcal{O}$ provides a geometric method to locate zeros of f without solving equations directly.

Lemma 29 (Zeros as discontinuities of Δz). A point z^* is a zero of f if and only if $\Delta z(r)$ is discontinuous at $r = |z^* - x_0|$:

$$z^* \text{ zero of } f \iff \lim_{r \rightarrow |z^* - x_0|^+} \Delta z(r) \neq \lim_{r \rightarrow |z^* - x_0|^-} \Delta z(r)$$

Zeros of f are the jump points of $\Delta z(r)$ as the orbital radius varies.

Algorithm (localisation of zeros in Ω):

1. Choose $P(x) = x - x_0$ with x_0 at centre of Ω
2. Compute $\Delta z(r)$ for decreasing radii r
3. Each jump in $\Delta z(r)$ signals a zero at radius r
4. Move x_0 and repeat — triangulate to find exact position

This is a geometric criterion, not an analytic one. No equation is solved. Zeros are located by observing how orbits change as the radius shrinks.

25 Orbital Spectroscopy

Classical analysis separates two problems: (1) how many zeros in Ω ? (Rouché's theorem, hard) (2) where are they? (Newton's method, iterative)

In $\mathcal{O}$, a single radial sweep gives both simultaneously.

Definition 2 (Orbital spectrum). *The orbital spectrum of f with respect to x_0 is:*

$$\mathcal{S}(f, x_0) = \{r > 0 : \Delta z(r^+) \neq \Delta z(r^-)\}$$

the set of discontinuities of $\Delta z(r)$.

Theorem 2 (Spectroscopy theorem).

$$|\mathcal{S}(f, x_0)| = N + P$$

The cardinality of the orbital spectrum equals the total number of zeros and poles of f in the plane, counted by distance from x_0 .

Each element $r^ \in \mathcal{S}(f, x_0)$ gives the distance from x_0 to a zero or pole of f . Two orbital spectra from distinct centres $x_0^{(1)}$ and $x_0^{(2)}$ locate each zero exactly by intersection of circles.*

Remark 5 (Structural advantage). $\mathcal{O}$ provides global information (how many zeros) before local information (where they are). Classical methods require knowing the count before searching. Orbital spectroscopy reverses this: count first, locate second.

26 Orbital Symmetry and the Critical Line

Applied to $f(x) = \zeta(x)$, the orbital spectrum from a real centre x_0 is:

$$\mathcal{S}(\zeta, x_0) = \{r_k = |x_0 - z_k| : z_k \text{ zero of } \zeta\}$$

Lemma 30 (Orbital symmetry under RH). *If all non-trivial zeros of ζ lie on $\text{Re}(s) = \frac{1}{2}$, then for every real x_0 :*

$$r_k(x_0) = r_k(1 - x_0)$$

The orbital spectra from x_0 and $1 - x_0$ are identical. Symmetry of the spectrum is equivalent to symmetry of zeros around the critical line.

Remark 6 (Spectrum from $x_0 = \frac{1}{2}$). *From the centre of the critical line, the orbital spectrum reduces to:*

$$r_k = \left| \frac{1}{2} - \left(\frac{1}{2} + it_k \right) \right| = t_k$$

The orbital spectrum from $x_0 = \frac{1}{2}$ is exactly the sequence of imaginary parts of the non-trivial zeros.

Remark 7 (Geometric translation of RH). *A zero $\frac{1}{2} + \epsilon + it$ off the critical line produces two distinct distances from any real x_0 : $\sqrt{(x_0 - \frac{1}{2} - \epsilon)^2 + t^2}$ and $\sqrt{(x_0 - \frac{1}{2} + \epsilon)^2 + t^2}$. The orbital symmetry $r_k(x_0) = r_k(1 - x_0)$ is broken.*

In the language of $\mathcal{O}$:

$$\text{RH} \iff r_k(x_0) = r_k(1 - x_0) \text{ for all real } x_0$$

This is a reformulation, not a proof.

27 Product Structure: A Commutative Monoid

Definition 3 (Product of orbital spaces). *The product of two orbital spaces with polynomials P_1 and P_2 is the orbital space with $P_1 + P_2$:*

$$\mathcal{O}_1 \cdot \mathcal{O}_2 : e^{(P_1+P_2)(az+b)} = f_1 \cdot f_2$$

Lemma 31 (Barycentre of boundaries). *For $P_k(x) = x - \alpha_k$, the boundary of the product of n spaces is:*

$$x^* = \frac{1}{n} \sum_{k=1}^n \alpha_k$$

The structural boundary of a product is the barycentre of the roots of all factor polynomials.

Theorem 3 (Commutative monoid). *The set of orbital spaces under the product operation forms a commutative monoid:*

- **Associative:** $(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$
- **Commutative:** $P_1 + P_2 = P_2 + P_1$
- **Identity:** $P_0 = 0$ (formal, degenerate)
- **No inverse:** $P + (-P) = 0$ gives $P = 0$, which is not a valid orbital space (division by zero in the orbital equation)

The identity element $P_0 = 0$ is formal — it does not correspond to a valid orbital space. The structure is a monoid, not a group.

28 Differential Calculus of $\mathcal{O}$

A point of $\mathcal{O}$ is a triple (a, b, x_0) in $\mathbb{R}^* \times \mathbb{R} \times \mathbb{C}$. Three natural directions of differentiation exist.

Definition 4 (Three orbital derivatives). *Recall the orbital solution:*

$$z(x) = \frac{1}{a} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

(1) Rigid derivative in b :

Differentiate directly with respect to b :

$$\frac{\partial z}{\partial b} = \frac{\partial}{\partial b} \left[\frac{1}{a} \left(\frac{\ln f}{P} - b \right) \right] = \frac{1}{a} \cdot (-1) = -\frac{1}{a}$$

This is a constant — independent of x and of the position on the orbit. The entire orbit translates at uniform speed $-1/a$ when b increases.

(2) Flexible derivative in a :

Differentiate with respect to a :

$$\frac{\partial z}{\partial a} = \frac{\partial}{\partial a} \left[\frac{1}{a} \right] \cdot \left(\frac{\ln f(x)}{P(x)} - b \right) = -\frac{1}{a^2} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

This depends on x — it varies along the orbit. At points where $\frac{\ln f}{P} = b$, the derivative vanishes: these are fixed points of the a -flow.

(3) Functional derivative in x_0 :

The orbit path is $x(\theta) = x_0 + re^{i\theta}$ and $P(x) = x - x_0$. Write $u = re^{i\theta}$ (independent of x_0). Then $P = u$ and $x = x_0 + u$:

$$z = \frac{1}{a} \left(\frac{\ln f(x_0 + u)}{u} - b \right)$$

Differentiating with respect to x_0 :

$$\frac{\partial z}{\partial x_0} = \frac{1}{a} \cdot \frac{1}{u} \cdot \frac{f'(x_0 + u)}{f(x_0 + u)} = \frac{1}{a} \cdot \frac{(\ln f)'(x)}{P(x)}$$

The logarithmic derivative of f divided by P — a functional of f evaluated on the orbit.

The three derivatives have distinct natures:

- $\partial_b z = -\frac{1}{a}$ scalar, constant
- $\partial_a z = -\frac{1}{a^2} \left(\frac{\ln f}{P} - b \right)$ function on orbit
- $\partial_{x_0} z = \frac{(\ln f)'}{a \cdot P}$ functional of f

Remark 8 (Applications). This differential structure enables:

- **Orbital flows:** families of orbits evolving along a derivative direction. The b -flow is rigid (parallel translation); the a -flow is flexible (dilation).
- **Spectral derivative:** $\partial_{x_0} \mathcal{S}(f, x_0)$ measures how detected zeros move as the centre shifts — a derivative of the orbital spectrum.
- **Orbital optimisation:** gradient descent in (a, b, x_0) to find orbits closest to zeros of f .
- **Orbital Laplacian:** combining the three directions into a second-order operator $\Delta_{\mathcal{O}}$ — eigenvalues open.

29 Spectral Derivative

The orbital spectrum $\mathcal{S}(f, x_0) = \{r_k\}$ where $r_k = |x_0 - z_k^*|$ are distances from x_0 to zeros z_k^* of f .

Derivation. When x_0 moves by ϵ in direction $\hat{n}$:

$$r_k(x_0 + \epsilon \hat{n}) = |x_0 + \epsilon \hat{n} - z_k^*|$$

Write $r_k^2 = |x_0 - z_k^*|^2 = (x_0 - z_k^*) \overline{(x_0 - z_k^*)}$. Differentiating:

$$2r_k \frac{\partial r_k}{\partial x_0} = \overline{(x_0 - z_k^*)} + (x_0 - z_k^*) = 2 \operatorname{Re}(x_0 - z_k^*)$$

So in the real direction:

$$\frac{\partial r_k}{\partial \operatorname{Re}(x_0)} = \frac{\operatorname{Re}(x_0 - z_k^*)}{r_k}$$

and in the imaginary direction:

$$\frac{\partial r_k}{\partial \text{Im}(x_0)} = \frac{\text{Im}(x_0 - z_k^*)}{r_k}$$

Combined as a complex gradient:

$$\nabla_{x_0} r_k = \frac{x_0 - z_k^*}{r_k}$$

This is the unit vector pointing from z_k^* toward x_0 .

Definition 5 (Spectral derivative).

$$\frac{\partial \mathcal{S}}{\partial x_0} = \left\{ \frac{x_0 - z_k^*}{r_k} \right\}_k$$

The set of unit vectors from each zero toward x_0 .

Lemma 32 (Spectral stationarity). *The sum of spectral derivatives vanishes at the barycentre of the zeros:*

$$x_0 = \frac{1}{N} \sum_k z_k^* \implies \sum_k \frac{x_0 - z_k^*}{r_k} \approx 0$$

The barycentre is the point of spectral stationarity — the centre where the orbital spectrum is least sensitive to displacement of x_0 .

30 Higher Derivatives and Relations

Derivative in x :

Differentiating $z = \frac{1}{a} \left(\frac{\ln f}{P} - b \right)$ with respect to x by the quotient rule:

$$\frac{\partial z}{\partial x} = \frac{1}{a} \cdot \frac{P \cdot f'/f - P' \cdot \ln f}{P^2}$$

For $P(x) = x - x_0$ (so $P' = 1$):

$$\frac{\partial z}{\partial x} = \frac{1}{a} \cdot \frac{(x - x_0) \cdot f'/f - \ln f}{(x - x_0)^2}$$

Zeros of $\partial z / \partial x$ satisfy the transcendental equation $(x - x_0) \cdot f'/f = \ln f$ — the *critical points* of the orbit.

Second derivative in b :

Since $\partial z / \partial b = -1/a$ is constant:

$$\frac{\partial^2 z}{\partial b^2} = 0$$

The b -flow is linear — no curvature, no acceleration.

Second derivative in a :

Differentiating $\partial z / \partial a = -\frac{1}{a^2} \left(\frac{\ln f}{P} - b \right)$:

$$\frac{\partial^2 z}{\partial a^2} = \frac{2}{a^3} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

Lemma 33 (Euler relation of the a -flow). *The function z satisfies the following ODE in a :*

$$a \frac{\partial^2 z}{\partial a^2} + 2 \frac{\partial z}{\partial a} = 0$$

Derivation: $a \cdot \frac{2}{a^3} \left(\frac{\ln f}{P} - b \right) + 2 \cdot \left(-\frac{1}{a^2} \right) \left(\frac{\ln f}{P} - b \right) = \frac{2}{a^2} \left(\frac{\ln f}{P} - b \right) - \frac{2}{a^2} \left(\frac{\ln f}{P} - b \right) = 0.$

The a -flow is not free — it satisfies an Euler-type constraint. The orbit cannot dilate arbitrarily.

Cross derivatives:

$$\frac{\partial^2 z}{\partial a \partial b} = \frac{\partial}{\partial a} \left(-\frac{1}{a} \right) = \frac{1}{a^2} \quad (\text{constant, symmetric})$$

$$\frac{\partial^2 z}{\partial b \partial x_0} = \frac{\partial}{\partial b} \left(\frac{(\ln f)'}{a \cdot P} \right) = 0$$

The b -flow and the x_0 -flow do not interact — rigid translation is independent of the centre.

31 Orbital Integrals

Integral of z over a full orbit:

$$\oint z d\theta = \int_0^{2\pi} \frac{1}{a} \left(\frac{\ln f(x_0 + re^{i\theta})}{re^{i\theta}} - b \right) d\theta$$

Splitting into two terms:

$$= \frac{1}{ar} \int_0^{2\pi} e^{-i\theta} \ln f(x_0 + re^{i\theta}) d\theta - \frac{2\pi b}{a}$$

The first term is the Fourier coefficient c_{-1} of $\ln f$ on the circle of radius r . By the Cauchy formula, if f has no zeros inside:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \ln f(x_0 + re^{i\theta}) d\theta = 0$$

Lemma 34 (Centre of mass of an orbit). *If f has no zeros inside the orbit:*

$$\oint z d\theta = -\frac{2\pi b}{a}$$

The centre of mass of the orbit is $-b/a$, independent of f and P .

If f has zeros inside, residues contribute and the centre of mass shifts — it detects the zeros of f .

Integral of z with respect to b :

From $\partial z / \partial b = -1/a$, integrating:

$$\int_{b_1}^{b_2} z db = \int_{b_1}^{b_2} \frac{1}{a} \left(\frac{\ln f}{P} - b \right) db$$

$$= \frac{1}{a} \left[\frac{\ln f}{P} (b_2 - b_1) - \frac{b_2^2 - b_1^2}{2} \right]$$

Setting $\Delta b = b_2 - b_1$ and $\bar{b} = (b_1 + b_2)/2$:

$$\int_{b_1}^{b_2} z db = \frac{\Delta b}{a} \left(\frac{\ln f}{P} - \bar{b} \right) = \Delta b \cdot z(a, \bar{b}, x_0)$$

Lemma 35 (Area swept by b -flow). *The integral of z over a b -interval equals Δb times the orbit at the midpoint $\bar{b}$:*

$$\int_{b_1}^{b_2} z db = \Delta b \cdot z(a, \bar{b}, x_0)$$

The area swept by the rigid flow is exactly the orbit at the centre of the interval, scaled by Δb . This is the mean value theorem of $\mathcal{O}$: the b -flow is so uniform that the mean equals the midpoint value exactly.

Integral of z with respect to r :

Since $x(\theta) = x_0 + re^{i\theta}$, both f and P depend on r . Writing the main term with integration by parts ($u = \ln f$, $dv = dr/r$):

$$\int_{r_1}^{r_2} \frac{\ln f(x_0 + re^{i\theta})}{re^{i\theta}} dr = e^{-i\theta} \left[\ln f \cdot \ln r \right]_{r_1}^{r_2} - \int_{r_1}^{r_2} (\ln f)'(x) \cdot \ln r dr$$

where $(\ln f)' = f'/f$ evaluated at $x = x_0 + re^{i\theta}$. The full integral:

$$\int_{r_1}^{r_2} z dr = \frac{e^{-i\theta}}{a} \left[\ln f \cdot \ln r \right]_{r_1}^{r_2} - \frac{1}{a} \int_{r_1}^{r_2} (\ln f)'(x) \cdot \ln r dr - \frac{b(r_2 - r_1)}{a}$$

Lemma 36 (Double logarithm of the r -integral). *The r -integral produces a double logarithm $\ln f \cdot \ln r$ — the logarithm of the function times the logarithm of the radius. It does not simplify in general: the residual $\int (\ln f)' \cdot \ln r dr$ depends on the full structure of f .*

Special case $f(x) = x^n$: $\ln f = n \ln(x_0 + re^{i\theta})$ and the integral becomes explicit.

Summary of orbital integrals:

Integral	Result	Nature
$\oint z d\theta$	$-2\pi b/a$ (no zeros inside)	Fourier / Cauchy
$\int z db$	$\Delta b \cdot z(\bar{b})$	Linear (mean value)
$\int z da$	$az \cdot \ln(a_2/a_1)$	Logarithmic
$\int z dr$	$\frac{1}{a} [\ln f \cdot \ln r] - \dots$	Double logarithm

Each parameter produces a qualitatively different integral — Fourier, linear, logarithmic, double logarithmic. The calculus of $\mathcal{O}$ is not uniform: each direction has its own algebraic nature.

32 The Orbital Laplacian

Second derivative in x_0 :

From $\partial z / \partial x_0 = (\ln f)'(x) / (a \cdot P)$, differentiating again with $x = x_0 + u$, $P = u$ fixed:

$$\frac{\partial^2 z}{\partial x_0^2} = \frac{(\ln f)''(x)}{a \cdot P(x)}$$

Dimensional analysis. The three parameters have dimensions: a, b dimensionless [1]; $x_0 \in \mathbb{C}$ has dimension $[X]$. The second derivatives have dimensions:

$$\left[\frac{\partial^2 z}{\partial a^2} \right] = [Z], \quad \left[\frac{\partial^2 z}{\partial b^2} \right] = [Z], \quad \left[\frac{\partial^2 z}{\partial x_0^2} \right] = \frac{[Z]}{[X]^2}$$

The only characteristic length in $\mathcal{O}$ is the orbital radius r . Imposing dimensional homogeneity:

Definition 6 (Orbital Laplacian).

$$\Delta_{\mathcal{O}} = \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + r^2 \frac{\partial^2}{\partial x_0^2}$$

Applied to z , using $\partial^2 z / \partial b^2 = 0$:

$$\Delta_{\mathcal{O}} z = \frac{2}{a^3} \left(\frac{\ln f}{P} - b \right) + \frac{r^2 (\ln f)''(x)}{a \cdot P(x)}$$

Definition 7 (Harmonically orbital functions). A function f is harmonically orbital if $\Delta_{\mathcal{O}} z = 0$, i.e.:

$$(\ln f)''(x) = -\frac{2az \cdot P(x)}{r^2}$$

This is the equation of harmonic orbits — the functions f whose orbit is a fixed point of the orbital Laplacian.

Remark 9 (Commutativity). The orbital Laplacian commutes with the a -flow: $[\Delta_{\mathcal{O}}, \partial / \partial a] = 0$ for all choices of coefficients. This follows from direct computation and holds independently of the dimensional choice of r^2 . The Euler relation $a \partial_a^2 z + 2 \partial_a z = 0$ is a separate constraint on the a -flow, not imposed by the Laplacian.

33 Eigenvalues of the Orbital Laplacian

We seek z such that $\Delta_{\mathcal{O}} z = \lambda z$. Substituting and multiplying by a :

$$\frac{2Q}{a^2} + \frac{r^2 (\ln f)''}{P} = \lambda Q \quad Q = \frac{\ln f}{P} - b = az$$

Isolating $(\ln f)''$:

$$(\ln f)'' = \frac{P}{r^2} \left(\lambda - \frac{2}{a^2} \right) \left(\frac{\ln f}{P} - b \right)$$

Eigenvalue $\lambda = 2/a^2$:

The right side vanishes, giving $(\ln f)'' = 0$, so:

$$\ln f(x) = \alpha x + \beta \implies f(x) = e^{\alpha x + \beta}$$

Theorem 4 (Continuous spectrum). *Exponential functions $f(x) = e^{\alpha x + \beta}$ are eigenfunctions of $\Delta_{\mathcal{O}}$ with eigenvalue:*

$$\lambda = \frac{2}{a^2}$$

This eigenvalue depends on a — it varies continuously over $\mathbb{R}_{>0}$. The orbital Laplacian has a continuous spectrum parameterised by a .

Since $\lambda = 2/a^2 > 0$ for all $a \in \mathbb{R}^$, the continuous spectrum is strictly positive: $\Delta_{\mathcal{O}}$ has no negative eigenvalues in the continuous part.*

Remark 10 (Discrete spectrum). *Under the periodicity constraint $z(\theta + 2\pi) = z(\theta)$ — enforced by the Euler condition — the spectrum may acquire a discrete part. This requires f to satisfy a monodromy condition around the orbit. The discrete spectrum of $\Delta_{\mathcal{O}}$ is open.*

Remark 11 (Summary of the spectral structure). • *Continuous spectrum: $\lambda = 2/a^2$, eigenfunctions $f = e^{\alpha x + \beta}$*

- *Discrete spectrum: under periodicity — open*
- *Positivity: $\lambda > 0$ in the continuous part*
- *The spectrum depends on a — each value of a selects a different eigenvalue*

34 Discrete Spectrum: Periodicity and Quasi-Periodicity

Periodicity condition. After one full turn $\theta \rightarrow \theta + 2\pi$, since $e^{i(\theta+2\pi)} = e^{i\theta}$, the path $x(\theta)$ returns to its starting point. However $\ln f$ is multivalued in $\mathbb{C}$:

$$\ln f(x_0 + re^{i(\theta+2\pi)}) = \ln f(x_0 + re^{i\theta}) + 2\pi iN$$

where N is the number of zeros of f inside the orbit (winding number). The displacement of z :

$$\Delta z = z(\theta + 2\pi) - z(\theta) = \frac{2\pi iN}{a \cdot re^{i\theta}}$$

Lemma 37 (Strict vs quasi-periodicity). • *$N = 0$ (no zeros inside): $\Delta z = 0$, the orbit is strictly periodic*

- *$N \neq 0$: $\Delta z \neq 0$, the orbit is quasi-periodic — it shifts by $2\pi iN/(are^{i\theta})$ after each turn*

Quasi-periodicity quantifies the zeros of f inside the orbit. The winding number N is the obstruction to strict periodicity.

Discrete eigenvalue problem. For strict periodicity ($N = 0$), the eigenvalue equation $\Delta_{\mathcal{O}} z = \lambda z$ becomes:

$$(\ln f)'' = \frac{P(x)}{r^2} \left(\lambda - \frac{2}{a^2} \right) \left(\frac{\ln f}{P} - b \right)$$

with f having no zeros inside the circle of radius r . This is a Sturm-Liouville equation on $[0, 2\pi]$ with periodic boundary conditions. It admits a discrete spectrum.

Remark 12 (Discrete spectrum — open). *The full characterisation of the discrete spectrum — eigenvalues, eigenfunctions, completeness — requires solving the Sturm-Liouville problem above. This is open.*

35 Metric Tensor and Intrinsic Dimension

A displacement (da, db, dx_0) in parameter space produces:

$$dz = -\frac{Q}{a^2}da - \frac{1}{a}db + \frac{(\ln f)'}{aP}dx_0, \quad Q = az$$

The naive metric tensor from $|dz|^2$:

$$g = \begin{pmatrix} Q^2/a^4 & Q/a^3 & -Q(\ln f)'/(a^3P) \\ Q/a^3 & 1/a^2 & -(\ln f)'/(a^2P) \\ -Q(\ln f)'/(a^3P) & -(\ln f)'/(a^2P) & |(\ln f)'|^2/(a^2|P|^2) \end{pmatrix}$$

Lemma 38 (Rank-1 degeneracy). *The tensor g has rank 1. All rows are proportional to the gradient of z :*

$$\nabla z = \left(-\frac{Q}{a^2}, -\frac{1}{a}, \frac{(\ln f)'}{aP} \right)$$

Consequence: $\det g = 0$. *The parameter space (a, b, x_0) is not intrinsically three-dimensional for z — the three parameters are bound by:*

$$az = \frac{\ln f(x)}{P(x)} - b$$

a single equation. The effective dimension of $\mathcal{O}$ is 1, parameterised by the single invariant:

$$Q = az = \frac{\ln f}{P} - b$$

Remark 13 (Correct metric tensor). *The metric tensor in parameter space is degenerate. The correct metric of $\mathcal{O}$ is the Hausdorff distance between orbits — not a tensor in (a, b, x_0) . The tensor approach confirms that Hausdorff is the natural metric from the beginning.*

The intrinsic geometry of $\mathcal{O}$ lives in the space of orbits (closed curves in $\mathbb{C}$), not in the space of parameters.

36 Connection and Curvature

Since the effective dimension of $\mathcal{O}$ is 1 with intrinsic coordinate $Q = az$, the metric is:

$$|dz|^2 = \frac{|dQ|^2}{a^2} \implies g_{QQ} = \frac{1}{a^2}$$

Christoffel symbol:

$$\Gamma = \frac{1}{2}g^{QQ}\partial_Q g_{QQ} = \frac{a^2}{2}\partial_Q \left(\frac{1}{a^2} \right) = 0$$

since a and Q are independent.

Theorem 5 (Flatness of $\mathcal{O}$). *The intrinsic connection of $\mathcal{O}$ is flat: $\Gamma = 0$. The geodesics are the curves $Q = az = \text{const}$ — orbits of the a -flow.*

The three flows have distinct curvature natures:

- *b-flow: flat, rigid translation, trivial parallel transport*
- *a-flow: geodesics of $\mathcal{O}$, Q preserved along flow lines*
- *x_0 -flow: deformation by f , curvature lives in the space of orbits (Hausdorff), not in parameter space*

The apparent complexity of $\mathcal{O}$ in parameter space (a, b, x_0) is a coordinate artefact. In the intrinsic coordinate Q , the space is flat.

Remark 14 (Curvature from f). *The curvature of $\mathcal{O}$ is not intrinsic — it is induced by f through the x_0 -flow. Different choices of f give different Hausdorff geometries on the space of orbits. f is the source of extrinsic curvature. The intrinsic geometry is always flat.*

37 Admissible Orbits and Validity of the Hausdorff Metric

The Hausdorff metric is well-defined only when orbits are proper — bounded, continuous, without branch cuts. Three conditions can fail:

1. **Zeros of f on the orbit circle:** $\ln f$ diverges, the orbit is unbounded.
2. **Branch cuts of $\ln f$:** the orbit is discontinuous.
3. **Multiplicity blindness:** Hausdorff does not distinguish orbits that partially overlap with different winding structure.

Definition 8 (Admissible orbit). *An orbit (a, b, x_0, r) is admissible if:*

$$f(x) \neq 0 \quad \text{for all } x = x_0 + re^{i\theta}, \quad \theta \in [0, 2\pi]$$

No zero of f lies on the circle of radius r centred at x_0 .

Lemma 39 (Hausdorff is a metric on admissible orbits). *For two admissible orbits C_1, C_2 : $d_H(C_1, C_2)$ is finite, well-defined, and satisfies all metric axioms. The space of admissible orbits equipped with d_H is a metric space.*

Remark 15 (Relation to the founding condition). *The admissibility condition extends the founding condition $\text{roots}(P) \cap \text{roots}(f) = \emptyset$ from the centre x_0 to the full orbit circle. It is the natural domain of validity of $\mathcal{O}$. Orbits that touch a zero of f are boundary objects — they mark the transition between admissible regions.*

The intrinsic coordinate $Q = az = \frac{\ln f}{P} - b$ has a natural interpretation.

Definition 9 (Relative complexity). *Q measures the complexity of f relative to P : how much $\ln f$ exceeds a linear dependence on P , shifted by b .*

- $f = e^P$: $Q = 1 - b$ — constant, f has exactly the complexity of P
- $f = e^{P^2}$: $Q = P - b$ — linear in x , f exceeds P by one degree
- $f = e^{e^P}$: $Q = e^P - b$ — exponential, f vastly exceeds P

Geodesics $Q = \text{const}$ are orbits where the relative complexity of f with respect to P is fixed.

Summary of Established Results

Result	Statement	Status
Orbital equation	$e^{P(az+b)} = f$	Established
Mixed solutions	z real or complex by sign of f	Established
Foliation	Non-uniform, branches never cross	Established
Euler constraint	= metric between leaves	Established
Symmetries	Dihedral group D_∞	Established
Classification	$(P, f, \deg g)$, 2^{n+1} classes	Established
Encoding theorem	N, P readable from orbits	Established
Orbital spectrum	$ \mathcal{S} = N + P$	Established
Mean value	$\int z db = \Delta b \cdot z(\bar{b})$	Established
a -invariant	$az = Q$ constant along a -flow	Established
Euler relation	$a\partial_a^2 z + 2\partial_a z = 0$	Established
Monoid	Product of spaces, commutative	Established
Flat connection	$\Gamma = 0$, geodesics $Q = \text{const}$	Established
Rank-1 tensor	Effective dimension 1	Established
Continuous spectrum	$\lambda = 2/a^2$, $f = e^{\alpha x}$	Established
Hausdorff metric	Full consequences	Open
Discrete spectrum	Sturm-Liouville problem	Open
Orbital Laplacian	Discrete eigenvalues	Open
Two-level topology	Full characterisation	Open

Since $\frac{\ln f}{P} - b$ is independent of a :

$$\int_{a_1}^{a_2} z da = \left(\frac{\ln f}{P} - b \right) \int_{a_1}^{a_2} \frac{da}{a} = \left(\frac{\ln f}{P} - b \right) \ln \frac{a_2}{a_1}$$

Recognising that $a \cdot z = \frac{\ln f}{P} - b$:

$$\int_{a_1}^{a_2} z da = a_1 z_1 \cdot \ln \frac{a_2}{a_1} = a_2 z_2 \cdot \ln \frac{a_2}{a_1}$$

Lemma 40 (Invariant of the a -flow and logarithmic integral). *The quantity az is an invariant of the a -flow:*

$$a_1 z(a_1) = a_2 z(a_2) = \frac{\ln f(x)}{P(x)} - b$$

The integral of the flexible flow is logarithmic — not linear. The natural logarithm appears in $\mathcal{O}$ without being imposed.

Comparison of the two flows:

- b -flow: $\int z db = \Delta b \cdot z(\bar{b})$ *linear*
- a -flow: $\int z da = az \cdot \ln(a_2/a_1)$ *logarithmic*

In Construction

- Hausdorff metric: full consequences — open
- Two-level topology: full characterisation — open

- Orbital Laplacian: eigenvalues — open
- Orbital flows: classification — open

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